University of El Oued
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2M Topo

Master 2/2025 – 2026

Correction of Regular session Exam in Topology

Ex01: $E = \mathcal{C}([0,1], \mathbb{R})$ equipped with $\|\cdot\|_\infty$

$$u_n : E \longrightarrow \mathbb{R}$$
$$f \longmapsto u_n(f) = n\left(f(\tfrac{1}{n}) - f(0)\right);\ \forall n \geq 1$$

1) Linearity: $\forall f, g \in E, \forall \alpha, \beta \in \mathbb{R};\ \forall n \geq 1$

$$u_n(\alpha f + \beta g) = n\left(\alpha f(\tfrac{1}{n}) + \beta g(\tfrac{1}{n}) - \alpha f(0) - \beta g(0)\right)$$
$$= \alpha n\left(f(\tfrac{1}{n}) - f(0)\right) + \beta n\left(g(\tfrac{1}{n}) - g(0)\right)$$
$$= \alpha u_n(f) + \beta u_n(g) \quad 01$$

Continuity:

$$\forall n \geq 1,\ |u_n(f)| \leq n\left(|f(\tfrac{1}{n})| + |f(0)|\right) \leq 2n\|f\|_\infty$$

Then $\forall n \geq 1;\ u_n \in E^*$ $\quad 01$

2) $\lim_{n \to +\infty} u_n(f) = \lim_{n \to +\infty} n\left(f(\tfrac{1}{n}) - f(0)\right) = \lim_{n \to +\infty} \dfrac{f(\tfrac{1}{n}) - f(0)}{\tfrac{1}{n}} = f'(0)$ $\quad 1$

Ex02: P is seminorm on E.

$$A = \{x \in E;\ P(x) < 1\}$$

A is convex iff $\forall x, y \in A;\ \forall \alpha \in [0,1];\ \alpha x + (1-\alpha) y \in A$

Let $x, y \in A$ and $\alpha \in [0,1]$, we have

$$P(\alpha x + (1-\alpha) y) \leq \alpha P(x) + (1-\alpha) P(y) < \alpha + (1-\alpha) = 1 \quad 01{,}5$$

then A is convex

A is balanced iff; $\forall \lambda;\ |\lambda| \leq 1 \Rightarrow \lambda A \subset A$

Let $\lambda;\ |\lambda| \leq 1$ and $x \in A$, then we have

$$P(\lambda x) = |\lambda| P(x) \leq P(x) < 1$$

this implies that $\lambda x \in A$, Thus $\lambda A \subset A$ $\quad 01{,}5$

A is absorbing iff $\forall x \in E\ \exists k>0\ \forall \lambda;\ |\lambda| \le k \Rightarrow \lambda x \in A$

Let $x \in E$, we take $k = \frac{1}{1+P(x)} > 0$, then (01)

$\forall \lambda;\ |\lambda| \le k;\ P(\lambda x) = |\lambda| P(x) \le k\,P(x) = \frac{P(x)}{1+P(x)} < 1$ (01)

Therefore; $\lambda x \in A$. (A is absorbing).

<u>EXo3:</u> $f_n : E \to \mathbb{R}$, $c>0$, $0<\alpha\le 1$

$$|f_n(x) - f_n(y)| \le C(d(x,y))^{\alpha};\ \forall n \in \mathbb{N},\ f_n(p) = 0$$

$\forall n \in \mathbb{N};\ \forall x \in E;$ we have

$$|f_n(x)| = |f_n(x) - f_n(p) + f_n(p)| \le |f_n(x) - f_n(p)| + |f_n(p)|$$

$$\text{(01)} \le |f_n(x) - f_n(p)| \le C(d(x,p))^{\alpha}$$

Since E is compact, then $\forall \varepsilon>0;\ E = \bigcup_{i=1}^{k} B(a_i, \varepsilon)$. Moreover,

$$d(x,p) \le d(x, a_{i_x}) + d(a_{i_x}, a_{i_p}) + d(a_{i_p}, p)$$

$$< \varepsilon + \max_{i,j=\overline{1,k}} d(a_i, a_j) + \varepsilon = 2\varepsilon + M;\quad M = \max_{i,j=\overline{1,k}} d(a_i,a_j)$$ (01)

Then, $|f_n(x)| \le C(2\varepsilon + M)^{\alpha}$. $(f_n)_n$ is uniformly bounded

In another side; we have

$$\forall x,y \in E,\ \forall n \in \mathbb{N};\ |f_n(x) - f_n(y)| \le C(d(x,y))^{\alpha} < \varepsilon$$

$$\Rightarrow (d(x,y))^{\alpha} < \frac{\varepsilon}{C}$$ (01)

$$\Rightarrow d(x,y) < \sqrt[\alpha]{\frac{\varepsilon}{C}}$$

Then $(f_n)_n$ is equicontinuous, when we take $\delta = \sqrt[\alpha]{\frac{\varepsilon}{C}}$ (01)

According to Ascoli's theorem, $(f_n)_n$ has a uniformly convergent subsequence $(f_{n_k})_k$

Since $(f_{n_k})_k$ is equicontinuous, then its limit f is continuous.

$$|f(p)| \le |f(p) - f_{n_k}(p)| + |f_{n_k}(p)| = |f(p) - f_{n_k}(p)| \xrightarrow[k\to+\infty]{} 0$$ (01)

then $f(p) = 0$. In addition, $|f_{n_k}(x) - f_{n_k}(y)| \le C(d(x,y))^{\alpha}$.

when $k \to +\infty$ $|f_{n_k}(x) - f_{n_k}(y)| \longrightarrow |f(x) - f(y)| \le C(d(x,y))^{\alpha}$ (01)

EX04: $\forall n$; $f_n(0)=f_n'(0)=0$, $|f_n''(x)|\leq 1$; $\forall x\in[0,1]$.

we will prove that $(f_n)_n$ verifies the condition of Ascoli Theorem.

$([0,1],|.|)$ is compact (It is clear). 01

$\forall x\in[0,1]$; $|f_n(x)|\leq|f_n(x)-f_n(0)|+|f_n(0)|\leq|f_n(x)-f_n(0)|$

$|f_n(x)-f_n(0)|=|f_n'(\xi)||x|\leq|f_n'(\xi)|$. 01

In another side;

$|f_n'(x)-f_n'(0)|\leq|f_n''(\eta)||x|\leq 1$

$\Rightarrow |f_n'(x)|\leq 1$; $\forall x\in[0,1]$. 01

Hence, $|f_n(x)-f_n(0)|\leq|f_n'(\xi)|\leq 1$.

This implies that; $|f_n(x)|\leq 1$, $\forall x\in[0,1]$; $\forall n\in\mathbb{N}$,

then $(f_n)_n$ is uniformly bounded.

Now, we prove the equicontinuity of $(f_n)_n$;

$\forall x,y\in[0,1]$; $\forall n\in\mathbb{N}$, we have

$|f_n(x)-f_n(y)|=|f_n'(\xi)||x-y|\overset{01}{\leq}|x-y|$. (Because, $|f_n'(x)|\leq 1,\forall x$)

If we take $\delta\overset{01}{=}\varepsilon>0$, $(f_n)_n$ is equicontinuous

Therefore, $(f_n)_n$ is verified the conditions of Ascoli's Theorem,

then $(f_n)_n$ has a uniformly convergent subsequence on $[0,1]$. 01