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Correction of Integral Equations Exam

Exercise 1

$$\varphi(x) = x + \lambda \int_0^1 (1 - 3xt)\varphi(t)dt$$

1. *Let*

$$\varphi(x) = x + \lambda \int_0^1 \varphi(t)dt - 3\lambda x \int_0^1 t\varphi(t)dt. \qquad (1)$$

Put $c_1 = \int_0^1 \varphi(t)dt$ *and* $c_1 = \int_0^1 \varphi(t)dt$ *we get*

$$\varphi(x) = x + \lambda c_1 - 3\lambda x c_2 = \lambda c_1 + (1 - 3\lambda c_2)x. \quad ....(1.5) \qquad (2)$$

2. $c_1 = \int_0^1 (\lambda c_1 + (1 - 3\lambda c_2)t)dt = \lambda c_1 + \frac{1-3\lambda c_2}{2}$.
So we get

$$2(1-\lambda)c_1 + 3\lambda c_2 - 1 = 0.$$

$$c_2 = \int_0^1 t(\lambda c_1 + (1 - 3\lambda c_2)t)dt = \frac{\lambda c_1}{2} + \frac{1 - 3\lambda c_2}{3}.$$

Then we get the following system

$$\begin{cases} 2(1-\lambda)c_1 + 3\lambda c_2 - 1 = 0 \\ 3\lambda c_1 - 6(1+\lambda)c_2 + 2 = 0. \end{cases} \quad ....(1)$$

$$\Delta = \begin{vmatrix} 2(1-\lambda) & 3\lambda \\ 3\lambda & -6(1+\lambda) \end{vmatrix} = 3(\lambda^2 - 4). \quad ....(1,5)$$

If $\lambda \neq 2$ *and* $\lambda \neq -2$, *the equation* (1) *has a solution, while if* $\lambda = 2$ *or* $\lambda = -2$ *there is no solution.* ....*(0.5)*

3. *If* $\lambda \neq 2$ *and* $\lambda \neq -2$, *then we get*

$$c_1 = \begin{vmatrix} 3\lambda & -1 \\ -6(1+\lambda) & 2 \end{vmatrix} = -\frac{2}{\lambda^2 - 4}. \quad ....(1)$$

$$c_2 = \begin{vmatrix} -1 & 2(1-\lambda) \\ 2 & 3\lambda \end{vmatrix} = \frac{\lambda - 4}{\lambda^2 - 4}. \quad ....1$$

Substituting in (2) *we get :*

$$\varphi(x) = \frac{1}{\lambda^2 - 4}\left(-2\lambda + (1 - 3\lambda x)(\lambda - 4)\right). \quad ....(1)$$

Exercise 2 *1.* $\|K\|_E^2 = \int_0^1\int_0^1 x^2t^2dtdx = \frac{1}{9} \Rightarrow \|K\|_E = \frac{1}{3}$. ....(1)

Define an operator : $T: X \to X$ *such that*

$$T\varphi(x) = x + \lambda\int_0^1 xtdt.$$

We will prove that T *is a contraction and applying Banach fixed point theorem, in fact we have : :*................

$|T\varphi(x) - T\psi(x)|^2 \leq |\lambda|\|\varphi-\psi\|_X^2 \int_0^1 |K^2(x,t)|dt$ *(Cauchy-Schawrtz inequality) :*.........................*(1)*.

Integrating from 0 *to* 1 *we get :*

$$\int_0^1 |T\varphi(x) - T\psi(x)|^2dx \leq |\lambda|\int_0^1\int_0^1 |K^2(x,t)|dtdx\|\varphi-\psi\|_X,$$

which implies that :

$\|T\varphi - T\psi\|_X \leq |\lambda|\|K\|_E.\|\varphi-\psi\|$. :....................*(0.5)*

$If\, /\lambda| < 3$ *implies that* T *is a contraction, then the equation has a unique solution.* ....*(1)*

2. *For* $\lambda \neq 3$, $\varphi_1(x) = 1 + \frac{\lambda}{2}x$, $\varphi_2(x) = 1 + \frac{\lambda}{2}x + \frac{\lambda^2}{6}x$ $\varphi_3(x) = 1 + \frac{\lambda}{2}x + \frac{\lambda^2}{6}x + \frac{\lambda^2}{18}x$........................(1.5)

$$\varphi_n(x) = 1 + \frac{\lambda}{2}x + \frac{\lambda^2}{6}x + ... + \frac{\lambda^n}{2.3^{n-1}}x = 1 + \frac{3\lambda x}{2(3-\lambda}(1 - \frac{1}{3^n}).\Omega.....(1)$$

3.

$$\varphi(x) = \lim_{n\to\infty}\varphi_n(x) = 1 + \frac{3\lambda x}{2(3-\lambda)}.................(1)$$

Exercise 3 *1.*

$$\mathcal{L}(\varphi) = \mathcal{L}(\cos x) - \mathcal{L}(\sin x) + 2\mathcal{L}(cosx)\mathcal{L}(\varphi)$$

$$\mathcal{L}(\varphi) = \frac{p}{p^2+1} - \frac{1}{p^2+1} + \frac{2p}{p^2+1}\mathcal{L}(\varphi). \quad ....(1)$$

2.

$$(1 - \frac{2p}{p^2+1})\mathcal{L}(\varphi) = \frac{p-1}{p^2+1}$$

$$\mathcal{L}(\varphi) = \frac{1}{p-1} \Rightarrow \varphi(x) = e^x. \quad ....(1)$$

3.

$$\mathcal{L}(\varphi') = \mathcal{L}(\cos x) - \mathcal{L}(\sin x) + 2\mathcal{L}(\sin x)\mathcal{L}(\varphi')...............................0.5$$

$$p\mathcal{L}(\varphi) = \frac{p-1+2p\mathcal{L}(\varphi)}{p^2+1}..(0.5)$$

$$\Rightarrow (p^3-p)\mathcal{L}(\varphi) = p-1 \Rightarrow \mathcal{L}(\varphi) = \frac{1}{p} - \frac{1}{p+1}..(1)$$

$$\Rightarrow \varphi(x) = x - e^{-x}...(1)$$