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Department of Mathematics

First-Year Master's Degree in Mathematics
Date : 26/01/2026, . Duration: 1hour 30minites

Module Examination : General Topology

Exercice 1. (7pts)

1. Show that the space $(\mathbb{Q}, |.|)$ is not Baire space.
2. Apply Baire's theorem to show that $]0,1[$ is not countable.
3. Let (X, δ) be a Baire space, $(F_n)_{n\in\mathbb{N}}$ a sequence of closed subsets of X such that:

$$\bigcup_{n\in\mathbb{N}} F_n = X.$$

Show that open set $\bigcup_{n\in\mathbb{N}} \mathring{F}_n$ is dense in X.

<u>Hint:</u> We can introduce subset $B = X \setminus \bigcup_{n\in\mathbb{N}} \mathring{F}_n$ and subsets $B_n = B \bigcap F_n$.

Exercice 2.(7pts)

1. Let $a, b \in \mathbb{R}$ with $a < b$, and let $\mathcal{F} \subset C^1(]a,b[)$ such that $\{|f'(x)| : f \in \mathcal{F}, x \in]a,b[\}$ is bounded by M. Prove that $\mathcal{F}$ is equicontinuous.
2. Let (X, d) be a compact metric space and let $(f_n)_{n\in\mathbb{N}}$ be a sequence of functions in $C(X, \mathbb{R})$. Suppose that:
   (a) The sequence (f_n) is *equicontinuous* on X.
   (b) The sequence (f_n) converges *pointwise* to a function $f : X \to \mathbb{R}$.

   Prove that (f_n) converges *uniformly* to f on X.

Exercice 3.(6pts)

Using the usual topology, answer the following questions, with justification :

1. Let $A = [0,1]$. Is the set $[0, 0.5[$ open in A?
2. Let $B = \{0\} \cup \{\frac{1}{n} : n \in \mathbb{N}^*\}$. Is the singleton $\{0\}$ open in B?
3. Let $C = [0,2]$. Is the set $]1,2]$ open in C?
4. Let $n \in \mathbb{N}$. Is $\{n\}$ open in $\mathbb{N}$?

Good Luck

Model answers for the General Topology examintion

Exercice 1 (7 pts):

1/ Hence, $\mathbb{Q}$ is a countable, then $\mathbb{Q} = \{q_n / n \in \mathbb{N}\}$ 0,5

For each $n \in \mathbb{N}$, define $F_n = \{q_n\}$, then $\mathbb{Q} = \bigcup_{n\in\mathbb{N}} F_n$ 0,25

but: $\{F_n\}_{n\in\mathbb{N}}$ is a sequence of closed and $\mathring{F}_n = \phi : \forall n$ 0,25 + 0,25

and moreover, $\overset{\circ}{\overline{\bigcup_{n\in\mathbb{N}} F_n}} = \mathbb{Q} \neq \emptyset$. . . . 0,25

Therefore, $(\mathbb{Q}, |.|)$ is not a Baire space. 0,5

2/ Assume by contradiction, that the interval $]0,1[$ is countable.

Then $]0,1[= \{r_n\}_{n\in\mathbb{N}}$. For each $n\in\mathbb{N}$, define $F_n = \{r_n\}$ 0,25

then: $\{F_n\}_{n\in\mathbb{N}}$ is a sequence of closed, and $\mathring{F}_n = \phi : \forall n$; $\overset{\circ}{\overline{\bigcup_{n\in\mathbb{N}} F_n}} =]0,1[$ 0,25

it would follow that $]0,1[$ is not a Baire space. 0,25

However, $]0,1[$ is a open subset of $\mathbb{R}$, and $\mathbb{R}$ is a Baire space, 0,25

Baire's theorem implies that $]0,1[$ is a Baire space 0,5

This contradiction shows that $]0,1[$ is not countable. 0,5

3/ $B_n = B \cap F_n$, with $B = X - \bigcup_{n\in\mathbb{N}} \mathring{F}_n$;

then $\{B_n\}_{n\in\mathbb{N}}$ is a sequence of closed. 0,5

$\forall n \in \mathbb{N}$ $\mathring{B}_n = \overset{\circ}{\overline{\left(X \setminus \bigcup_{m=0}^{\infty} \mathring{F}_m\right)}} \cap \mathring{F}_n \subset (X \setminus \mathring{F}_n) \cap \mathring{F}_n = \phi$ 01

$\left[\overset{\circ}{\overline{X \setminus \bigcup_{m=0}^{\infty} \mathring{F}_m}} \subset X \setminus \bigcup_{m=0}^{\infty} \mathring{F}_m \subset X - \mathring{F}_n \right]$

therefore $\overset{\circ}{\overline{\bigcup_{n=0}^{\infty} B_n}} = \phi$, but $\bigcup_{n=0}^{\infty} B_n = B$ 0,5 + 0,5

Hence $\mathring{B} = \phi$,

so $\overline{\bigcup_{n\in\mathbb{N}} \mathring{F}_n} = X$. 0,5

Ex(02): (7pts):

1/- For any $f \in \mathcal{F}$, and any $x, y \in]a, b[$, the "Mean Value Theorem" guarantees, the exists of a "c" $\in]x, y[$; such that

01

$$f(x) - f(y) = f'(c)(x-y)$$

then : $|f(x) - f(y)| = |f'(c)| \cdot |x-y| \leq M|x-y|$. 01

For a given $\varepsilon > 0$, we can choose $\delta = \frac{\varepsilon}{M}$. If $|x-y| < \delta$ then $|f(x) - f(y)| < M\delta = \varepsilon$; $\forall f \in \mathcal{F}$. 0,5

Then, the family $\mathcal{F}$ is equicontinuous on $]a, b[$. 0,5

2/- . Fix, $\varepsilon > 0$, and let $x \in X$, then exists $\delta = \delta(x, \varepsilon) > 0$, such that $\forall y \in X$; $d(x,y) < \delta \Longrightarrow |f_n(x) - f_n(y)| < \frac{\varepsilon}{3}$; $\forall n \in \mathbb{N}$ ..... (*) 0,5

. Since X is compact, and $X = \bigcup_{x \in X} B(x, \delta(x, \varepsilon))$, then exists $x_1, \dots, x_m \in X$, such that $X = \bigcup_{i=1}^{m} B(x_i, \delta(x_i, \varepsilon))$ 0,5

. let $\delta_i = \delta(x_i, \varepsilon)$, $i = \overline{1,m}$.

. Since $f_n \longrightarrow f$ pointwise, then for each $i = 1, \dots, m$, there exists N_i, such that ; $\forall n \geq N_i$: $|f_n(x_i) - f(x_i)| < \frac{\varepsilon}{3}$

then i.e. $\forall n \geq \max_{1 \leq i \leq m} N_i$; $|f_n(x_i) - f(x_i)| < \frac{\varepsilon}{3}$: $\forall i = \overline{1,m}$. // ..... (**) 0,5

. Take any, $x \in X$; $\Rightarrow \exists i = 1 \dots m$; $x \in B(x_i, \delta_i)$; then,

$|f_n(x_i) - f_n(x)| < \frac{\varepsilon}{3}$. $\forall n \in \mathbb{N}$ ..... (***) 0,5

. Using (***), and $f_n \longrightarrow f$ pointwise (i.e $f_n(x) \to f(x)$; $f_n(x_i) \to f(x_i)$)

then $|f(x) - f(x_i)| \leq \frac{\varepsilon}{3}$ ..... (****) 0,5

Now, by (**), (***) and (****), and following inequality:

$$|f_n(x)-f(x)| \leq |f_n(x)-f_n(x_i)|+|f_n(x_i)-f(x_i)|+|f(x_i)-f(x)| \quad 0,5$$

We conclud that:

$$\forall n \geq \max_{1\leq i\leq m} N_i : \quad |f_n(x)-f(x)| \leq \varepsilon ; \quad \forall x \in X. \quad 0,5$$

therefore, the entire sequence $(f_n)_n$ converges uniformly to f on X

Ex(03): (7pts):

1/. Yes: $[0, 0,5[$ is open in A: 0,5

justification:

$[0, 0,5[= [0,1] \cap]-1, 0,5[= A \cap]-1, 0,5[$ 01

with $]-1, 0,5[$ open set in $\mathbb{R}$

2/. No: $\{0\}$ is not open in B: 0,5

Justification:

If $\{0\} = B \cap V$, with V is open set in $\mathbb{R}$.

then: $\forall n \in \mathbb{N}^*$; $\frac{1}{n} \notin V$ i.e $\frac{1}{n} \in V^c$ (V^c: closed set) 0,5

therefor $0 = \lim_{n\to+\infty} \frac{1}{n} \in V^c$, Hence $0 \notin V$

Thus 0 cannot be expressed as $V \cap B$. 0,5

3/. Yes, $]1,2]$ is open in C 0,5

justification: $]1,2] =]1,+\infty[\cap [0,2] =]-1,+\infty[\cap C$ 01

4/. Yes, $\{n\}$ is open in $\mathbb{N}$ 0,5

justification: $\{n\} =]n-\frac{1}{2}, n+\frac{1}{2}[\cap \mathbb{N}$ 01