الحلول النموذجية ...

سلم تصحيح الامتحان :

التمرين الأول :

1) $x_i = \frac{\pi}{4} i$, $f(x_i) = \cos(x_i)$

x_i	$x_0=0$	$x_1=\frac{\pi}{4}$	$x_2=\frac{\pi}{2}$	$x_3=\frac{3\pi}{4}$
$f(x_i)$	1	$\sqrt{2}/2$	0	$-\sqrt{2}/2$

2) كثير حدود الاستكمال

عند النقاط x_0, x_1, x_2 لاغرانج

$$P_2(x) = L_0(x) f(x_0) + L_1(x) f(x_1) + L_2(x) f(x_2)$$

بما أن $f(x_2) = 0$ نجد

$$P_2(x) = L_0(x) f(x_0) + L_1(x) f(x_1)$$ (0.1)

$$L_0(x) = \frac{(x-x_1)(x-x_2)}{(x_0-x_1)(x_0-x_2)} = \frac{(x-\pi/4)(x-\pi/2)}{(-\frac{\pi}{4})(-\frac{\pi}{2})}$$

$$\boxed{L_0(x) = \frac{8}{\pi^2}x^2 - \frac{6}{\pi}x + 1}$$ (0.1)

$$L_1(x) = \frac{(x-x_0)(x-x_2)}{(x_1-x_0)(x_1-x_2)} = \frac{x(x-\pi/2)}{\frac{\pi}{4}(\frac{\pi}{4}-\frac{\pi}{2})}$$

$$\boxed{L_1(x) = -\frac{16}{\pi^2}x^2 + \frac{8}{\pi}x}$$ (0.2)

ومنه

$$P_2(x) = \frac{8}{\pi^2}x^2 - \frac{6}{\pi}x + 1 - \frac{8\sqrt{2}}{\pi^2}x^2 + \frac{4\sqrt{2}}{\pi}x$$

$$P_2(x) = \frac{8}{\pi^2}(1-\sqrt{2})x^2 - \frac{2}{\pi}(3-2\sqrt{2})x + 1$$ (0.2)

3) جدول الفروق المقسمة

x_i	$f[x_i]$	$f[x_i, x_{i+1}]$	$f[x_i, x_{i+1}, x_{i+2}]$	$f[x_i, x_{i+1}, x_{i+2}, x_{i+3}]$
0	1	$\frac{\frac{\sqrt{2}}{2}-1}{\frac{\pi}{4}} = \frac{2\sqrt{2}-4}{\pi}$ (0.1)	$\frac{-\frac{2\sqrt{2}}{\pi} - \frac{2\sqrt{2}-4}{\pi}}{\frac{\pi}{2}} = \frac{-8(\sqrt{2}-1)}{\pi^2}$ (0.2)	$\frac{-32(\sqrt{2}-1)}{3\pi^3}$ (0.2)
$\frac{\pi}{4}$	$\frac{\sqrt{2}}{2}$	$\frac{0-\frac{\sqrt{2}}{2}}{\pi/4} = -\frac{2\sqrt{2}}{\pi}$ (0.1)	0 (0.2)	
$\frac{\pi}{2}$	0	$\frac{-\frac{\sqrt{2}}{2}-0}{\frac{\pi}{4}} = -\frac{2\sqrt{2}}{\pi}$ (0.1)		
$\frac{3\pi}{4}$	$-\frac{\sqrt{2}}{2}$			

4) كثير حدود استيفاء نيوتن

عند $x_0 < x_1 < x_2 < x_3$ (0,5)

$P_3(x) = f(x_3) + (x-x_3) f[x_3, x_1]$

$+ (x-x_3)(x-x_1) f[x_3, x_1, x_2]$

$+ (x-x_3)(x-x_1)(x-x_2) f[x_3, x_1, x_2, x_0]$

(1) $P_3(x) = 1 + x \frac{2(\sqrt{2}-2)}{\pi} + x(x-\frac{\pi}{4})$

$\left(\frac{-8(\sqrt{2}-1)}{\pi^2}\right) + x(x-\frac{\pi}{4})(x-\frac{\pi}{2})$

$\left(\frac{-32(\sqrt{2}-1)}{3\pi^3}\right)$ (0,5) (0,25)

5) العلاقة بين P_3, P_2 هي

(0,25) $P_3(x) = P_2(x) + (x-\frac{\pi}{4})(x-\frac{\pi}{2}) \ldots$ (0,25)

6) حساب الخطأ $E_3(x)$

$E_3(x) = \frac{\max |f^{(4)}(x)|}{4!} \left|\prod_{i=0}^{4}(x-x_i)\right|$ (0,25)

(1) $f^{(4)}(x) = \cos x$ في $f(x) = \cos x$ لدينا

(0,25) $\max |f^{(4)}(x)| \leq \max |\cos x| \leq 1$

$E_3(x) = \frac{1}{24} \left|x(x-\frac{\pi}{4})(x-\frac{\pi}{2})(x-\frac{3\pi}{4})\right|$

4) حساب $E_3(\frac{\pi}{6})$ نجد

(0,25) $E_3(\frac{\pi}{6}) = \frac{1}{24} \left|\frac{\pi}{6}(\frac{\pi}{6}-\frac{\pi}{4})(\frac{\pi}{6}-\frac{\pi}{2})(\frac{\pi}{6}-\frac{3\pi}{4})\right|$ (0,25)

$= \frac{1}{24} \frac{\pi}{6} \left(\frac{\pi}{12}\right)\left(\frac{\pi}{3}\right)\left(\frac{7\pi}{12}\right)$

$= \frac{\pi^4}{24 \times 6 \times 12 \times 3 \times 12} = \frac{\pi^4}{\ldots}$

$\boxed{E_3(\frac{\pi}{6}) = 0,0016}$

(9)

التمرين الثاني:

نعتبر الدالة f التالية $f(x)=x^2-10$ معرفة على المجال $[3,4]$

حل المعادلة $f(x)=0$ هو $x=\sqrt{10}$

اختيار المجال $[a,b]$ التالي

$\sqrt{10}=3,16 \in [3,4]$

نتحقق من شروط تقارب نيوتن رافسون في المجال $[3,4]$.

1) $f(x)=x^2-10 \in \mathcal{C}^2([3,4])$

a) $f(3)=-1$, $f(4)=6$ $\Rightarrow f(3)\,f(4)<0$

2) $f'(x)=2x>0 \quad x\in[3,4]$

3) $f''(x)=2>0 \quad x\in[3,4]$

الدالة f تتقارب نحو $\sqrt{10}$.

اختيار x_0

$f(4)=6$, $f''(4)=2$ $\Rightarrow f(4)\,f''(4)>0$

الخوارزمية:

$$\begin{cases} x_0=4 \\ x_{n+1}=x_n-\dfrac{f(x_n)}{f'(x_n)} \end{cases}$$

$$\begin{cases} x_0=4 \\ x_{n+1}=x_n-\dfrac{x_n^2-10}{2x_n} \end{cases}$$

حساب:

n	x_n	$f(x_n)$	$f'(x_n)$	$f(x_n)/f'(x_n)$	$\lvert x_{n+1}-x_n\rvert$
0	4	6	8	0,75	
1	3,25	0,5625	6,5	0,086538	
2	3,163462	0,00749	6,326924	0,001183	
3	3,162279	0,000001	6,324558	0,000001	
4	3,162278				

$\lvert x_{n+1}-x_n\rvert = \left\lvert \dfrac{f(x_n)}{f'(x_n)} \right\rvert < \varepsilon$

من أجل $\boxed{n=4}$

إذن $\boxed{\sqrt{10}=3,162278\pm10^{-5}}$

$\max|f''(x)| = g(0) = g(2) = 6$

$n = \left[\sqrt{\frac{8\times 6}{6}}\right]+1 = [2,8]+1$ ومنه

$\boxed{n=3}$

2-2) حساب I_1 من أجل $n=3$

النقاط هي x_0, x_1, x_2, x_3

$x_0 = 0$, $x_1 = \frac{2}{3}$, $x_2 = \frac{4}{3}$, $x_3 = 2$

$h = \frac{b-a}{n} = \frac{2-0}{3} = \frac{2}{3}$ (1)

0	2/3	4/3	2
0	0,35	0,99	-4

$I_1 = \int_{x_0}^{x_3} f(x)\,dx =$

$\frac{h}{2}\left[f(x_0)+2f(x_1)+2f(x_2)+f(x_3)\right]$

$= \frac{1}{3}\left[0+0,7+1,98-4\right]$

$\boxed{I_1 = -1,76}$

2-3) الخطأ

$E = |I - I_1| = \left|-\frac{8}{5}+1,76\right| =$

$\boxed{E = 0,16}$ (1)

3-1) حساب I_5 بطريقة سمبسون

من أجل $n=5$ النقاط هي $x_0, x_1, x_2, x_3, x_4, x_5$ (1)

$x_0 = 0$, $x_5 = 2$

$h = \frac{b-a}{n} = \frac{2}{5} = 0,4$ (3)

التمرين الثالث

$f(x) = x^4 - 4x^3 + 3x^2$

1) حساب I

$I = \int_0^2 f(x)\,dx = \int_0^2 x^4 - 4x^3 + 3x^2\,dx$

$= \frac{1}{5}x^5 - x^4 + x^3 \Big|_0^2 = \frac{32}{5} - 16 + 8$

$= \frac{32}{5} - 8 = \frac{32-40}{5} = \boxed{\frac{-8}{5}}$ (1)

2-2) لدينا $\varepsilon = 0,1$ حساب عدد تقسيمات n بطريقة شبه المنحرف

$E(n) = \frac{(b-a)^3}{12n^2}M \le \varepsilon$ (1)

حيث $M = \max_{x\in[a,b]}|f''(x)|$

ومنه

$n = \left[\sqrt{\frac{(b-a)^3 M}{12\varepsilon}}\right]+1$

$n = \left[\sqrt{\frac{8}{12\times 0,1}M}\right]+1$

حساب M

$f(x) = x^4 - 4x^3 + 3x^2$

$f'(x) = 4x^3 - 12x^2 + 6x$

$f''(x) = 12x^2 - 24x + 6$

نضع $g(x) = |f''(x)|$

$g(x) = 12x^2 - 24x + 6$

$g'(x) = 24x - 24$

x	0		1		2
g'		$-$	0	$+$	
g	6	$\searrow$		$\nearrow$	6

$f^{(3)}(x) = 24x - 24$

$f^{(4)}(x) = 24$

$M = Max|f^{4}(x)| = 24$

$$n = \left[\sqrt{\frac{32 \times 24}{180 \times 10^{-5}}}\right] + 1$$

$$\boxed{n = 426670}$$

$x_0 = a = 0$

$x_1 = 0,4$, $x_2 = 0,8$, $x_3 = 1,2$ $x_4 = 1,6$

x	0	0,4	0,8	1,2	1,6	2
$f(x)$	0	0,26	0,28	-0,52	-2,15	-4

حساب I_5

$$I_5 = \int_{x_0}^{x_5} f(x)\,dx$$

$$= \int_{x_0}^{x_3} f(x)\,dx + \int_{x_3}^{x_5} f(x)\,dx$$

$$= \frac{3}{8} h \left[f(x_0) + 3f(x_1) + 3f(x_2) + f(x_3)\right]$$

$$+ \frac{h}{3}\left[f(x_3) + 4f(x_4) + f(x_5)\right]$$

$$= 0,15\left[0 + 0,78 + 0,84 - 0,52\right]$$

$$+ \frac{0,4}{3}\left[-0,52 - 8,6 - 4\right]$$

$$= 0,15(1,07) + \frac{0,4}{3}(-13,12)$$

$$= -1,587$$

ت 2-3 ، $\varepsilon = 10^{-5}$ أوجد

عدد التقسيمات اللازمة - سمسون

$E(x) =$

1

$$n^4 \geq \frac{(b-a)^5 M}{180\varepsilon} \quad / M = Max|f^{4}(x)|$$

$$n = \left[\sqrt[4]{\frac{(b-a)^5 M}{180\varepsilon}}\right] + 1$$

لدينا سابقا

$f''(x) = 12x^2 - 24x + 6$