University of El Oued
Faculty of Exact Sciences
Department of Mathematics

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Level : L2
Module : Introduction to topology

Exam
Sunday, 25 January 2026

Exercise 1. [7 points]

Consider $\mathbb{R}$ equipped with the distance

$$\forall x, y \in \mathbb{R}: \quad d(x,y) = |e^x - e^y|.$$

(1) Find $B(0,1)$ the open ball of center 0 and radius 1.
(2) Let $A = ]-\infty, 0]$. Prove that $A$ is bounded and closed in $(\mathbb{R}, d)$.
(3) Prove that $(\mathbb{R}, d)$ is not complete (indication : use the sequence $x_n = -n$, $n \in \mathbb{N}$).
(4) Prove that $A$ is not compact.

Exercise 2. [5 points]

(1) Consider

$$\forall x, y \in \mathbb{R}: \quad \delta(x,y) = \ln(1 + |x-y|).$$

Prove that $\delta$ is a distance on $\mathbb{R}$.

(2) Let $E = C([a,b], \mathbb{R})$ be the vector space of continuous functions on the closed and bounded interval $[a,b]$ equipped with the uniform norm $\|f\|_\infty = \sup_{x \in [a,b]} |f(x)|$, $f \in E$. Consider $\varphi : (E, \|.\|_\infty) \to (\mathbb{R}, |.|)$, $f \mapsto \varphi(f) = \int_a^b f(x)\,dx - f\Big(\frac{a+b}{2}\Big)$.
Prove that $\varphi$ is linear and continuous.

Exercise 3. [8 points]

Let $n \in \mathbb{N}$, $A_n = \{n, n+1, n+2, ...\}$ and

$$\begin{aligned} \tau &= \{\emptyset\} \cup \{A_n \,|\, n \in \mathbb{N}\} \\ &= \{\emptyset\} \cup \{A_0, A_1, A_2, ...\}. \end{aligned}$$

(1) Show that $\tau$ is a topology on $\mathbb{N}$.
(2) Let $B = \{1, 3, 5, 7, ...\}$. Find $\mathring{B}$ in $(\mathbb{N}, \tau)$.
(3) Find the closed sets in $(\mathbb{N}, \tau)$.
(4) Determine $\mathcal{N}(2)$ and $\mathcal{N}(3)$. Deduce that $(\mathbb{N}, \tau)$ is not Hausdorff.
(5) Prove that $(\mathbb{N}, \tau)$ is compact.

Exercise 1. [1.5+2.5+1.5+1.5]

(1) We have $B(0,1) = \{x \in \mathbb{R} |\, d(x,0) < 1\} = \{x \in \mathbb{R} |\, |e^x - \underbrace{e^0}_{=1}| < 1\}$

$$= \{x \in ]-\infty, 0[ \,|\, 1 - e^x < 1\} \cup \{x \in [0, \infty[ \,|\, e^x - 1 < 1\}$$
$$= \{x \in ]-\infty, 0[ \,|\, 0 < e^x\} \cup \{x \in [0, \infty[ \,|\, x < \ln 2\} = ]-\infty, 0[ \cup [0, \ln 2[ = ]-\infty, \ln 2[.$$

(2) A is bounded. We see that $A = ]-\infty, 0] \subset ]-\infty, \ln 2[ = B(0,1)$. Hence, $A$ is bounded in $(\mathbb{R}, d)$.
A is closed. Let $(x_n)_{n\in\mathbb{N}} \subset A$ be a sequence that converges in $(\mathbb{R}, d)$ to some limit $l \in \mathbb{R}$, that is $\lim_{n\to\infty} d(x_n, l) = \lim_{n\to\infty} |e^{x_n} - e^l| = 0$. This gives $\lim_{n\to\infty} e^{x_n} = e^l$ or $\lim_{n\to\infty} x_n = l$. Since $x_n \le 0$ for all $n \in \mathbb{N}$, it follows that $l = \lim_{n\to\infty} x_n \le 0$. Hence $l \in A$, and so $A$ is closed in $(\mathbb{R}, d)$.
(3) The sequence $x_n = -n$ is Cauchy in $(\mathbb{R}, d)$ because $\lim_{p,q\to\infty} d(x_p, x_q) = \lim_{p,q\to\infty} |e^{-p} - e^{-q}| = 0$. By absurd, suppose that $(-n)_{n\in\mathbb{N}}$ converges in $(\mathbb{R}, d)$ to some limit $l$. Namely, $\lim_{n\to\infty} d(-n, l) = \lim_{n\to\infty} |e^{-n} - e^l| = 0$. This gives $\underbrace{\lim_{n\to\infty} e^{-n}}_{=0} = e^l > 0$, which is absurd. Hence, the sequence $(-n)_{n\in\mathbb{N}}$ is divergent in $(\mathbb{R}, d)$, and so $(\mathbb{R}, d)$ is not complete.
(4) We see that $x_n = -n \in A$ for all $n \in \mathbb{N}$. By absurd, suppose that $(-n)_{n\in\mathbb{N}}$ has a subsequence $(-\varphi(n))_{n\in\mathbb{N}}$, where $\varphi : \mathbb{N} \to \mathbb{N}$ is a strictly increasing map, converging to some limit $l$. This leads to $\underbrace{\lim_{n\to\infty} e^{-\varphi(n)}}_{=0} = e^l > 0$, which is impossible. Hence, any subsequence of $(-n)_{n\in\mathbb{N}}$ is divergent, and so $A$ is not compact in $(\mathbb{R}, d)$.

Exercise 2. [3+2]

(1) (i) $\delta(x,y) = 0 \Leftrightarrow \ln(1 + |x-y|) = 0 \Leftrightarrow 1 + |x-y| = 1 \Leftrightarrow |x-y| = 0 \Leftrightarrow x = y$.
(ii) $\forall x, y \in \mathbb{R} :\quad \delta(x,y) = \ln(1 + |x-y|) = \ln(1 + |y-x|) = \delta(y,x)$, then $\delta$ is symmetry.
(iii) Triangle inequality. Let $x, y, z \in \mathbb{R}$. We have

$$\begin{aligned}
\delta(x,y) \le \delta(x,z) + \delta(z,y) &\Leftrightarrow \ln(1+|x-y|) \le \ln(1+|x-z|) + \ln(1+|z-y|)\\
&\Leftrightarrow e^{\ln(1+|x-y|)} \le e^{\ln(1+|x-z|)+\ln(1+|z-y|)}\\
&\Leftrightarrow e^{\ln(1+|x-y|)} \le e^{\ln(1+|x-z|)} e^{\ln(1+|z-y|)}\\
&\Leftrightarrow 1+|x-y| \le (1+|x-z|)(1+|z-y|)\\
&\Leftrightarrow \underbrace{|x-y| \le |x-z| + |z-y| + |x-z||z-y|}_{\text{it is true}}.
\end{aligned}$$

Hence, the triangle inequality holds. From (i), (ii) and (iii), $\delta$ is a distance on $\mathbb{R}$.
(2) $\varphi$ is linear. For all $f, g \in E$ and $\lambda \in \mathbb{K}$, we have

$$\begin{aligned}
\varphi(\lambda f + g) &= \int_a^b (\lambda f + g)(x)\, dx - (\lambda f + g)\Big(\frac{a+b}{2}\Big)\\
&= \int_a^b (\lambda f)(x) + g(x)\, dx - (\lambda f)\Big(\frac{a+b}{2}\Big) - g\Big(\frac{a+b}{2}\Big)\\
&= \int_a^b \lambda f(x)\, dx + \int_a^b g(x)\, dx - \lambda f\Big(\frac{a+b}{2}\Big) - g\Big(\frac{a+b}{2}\Big)\\
&= \lambda \int_a^b f(x)\, dx + \int_a^b g(x)\, dx - \lambda f\Big(\frac{a+b}{2}\Big) - g\Big(\frac{a+b}{2}\Big)\\
&= \lambda\Big[\int_a^b f(x)\, dx - f\Big(\frac{a+b}{2}\Big)\Big] + \int_a^b g(x)\, dx - g\Big(\frac{a+b}{2}\Big)\\
&= \lambda\varphi(f) + \varphi(g).
\end{aligned}$$

So, $\varphi$ is linear.

Continuity of $\varphi$. One can write

$$\begin{aligned}
\forall f \in E: \quad |\varphi(f)| &= \Big| \int_a^b f(x)\,dx - f\Big(\frac{a+b}{2}\Big) \Big| \\
&\leq \Big| \int_a^b f(x)\,dx \Big| + \Big| f\Big(\frac{a+b}{2}\Big) \Big| \\
&\leq \int_a^b |f(x)|\,dx + \Big| f\Big(\frac{a+b}{2}\Big) \Big| \\
&\leq \sup_{x\in[a,b]} |f(x)| \int_a^b dx + \sup_{x\in[a,b]} |f(x)| \\
&= (b-a)\|f\|_\infty + \|f\|_\infty \\
&\leq \underbrace{2\max(b-a,1)}_{C} \|f\|_\infty \ldots\ldots\ldots\ldots\ldots\ldots (*)
\end{aligned}$$

Since $\varphi$ is linear, it follows from $(*)$ that $\varphi$ is continuous on $E$.

Exercice 3.[2.5+1+2+1.5+1]

(1) (i) $\emptyset \in \tau$ and $\mathbb{N} = A_0 \in \tau$.

(ii) Let $B_1, B_2 \in \tau$. We distinguish two cases :

- $B_1 = \emptyset$ or $B_2 = \emptyset$, then $B_1 \cap B_2 = \emptyset \in \tau$.
- $B_1 \neq \emptyset$ and $B_2 \neq \emptyset$, then $B_1 = A_{n_1}$ and $B_2 = A_{n_2}$ with $n_1, n_2 \in \mathbb{N}$. Setting $n_3 = \max(n_1, n_2)$, we get $B_1 \cap B_2 = A_{n_3} \in \tau$.

(iii) Let $(B_i)_{i\in I} \subset \tau$. We distinguish two cases :

$*$ $\forall i \in I: \; B_i = \emptyset$, then $\bigcup\limits_{i\in I} B_i = \emptyset \in \tau$.

$*$ There exists $i \in I$ such that $B_i \neq \emptyset$. Therefore, $B_i = A_{n_i}$ for some $n_i \in \mathbb{N}$. Setting $m = \min\{n_i |\, B_i = A_{n_i}, i \in I, n_i \in \mathbb{N}\}$, we find $\bigcup\limits_{i\in I} B_i = A_m \in \tau$.

From (i), (ii) and (iii), $\tau$ is a topology on $\mathbb{N}$.

(2) If $m \in B = \{1, 2, 3, 7, ...\}$, then $m \notin \mathring{B}$ because there do not exist $n \in \mathbb{N}$ such that $m \in \underbrace{A_n}_{\in\tau} \subset B$. Hence, $\mathring{B} = \emptyset$.

(3) $B$ is closed $\Rightarrow \complement_{\mathbb{N}} B \in \tau \Rightarrow \complement_{\mathbb{N}} B = \emptyset$ or $\complement_{\mathbb{N}} B = A_n$, $n \in \mathbb{N} \Rightarrow B = \mathbb{N}$ or $B = \emptyset$ or $B = \{0, 1, ..., n\}$, $n \in \mathbb{N}$. If $B = \{0, 1, ..., n\}$ for some $n \in \mathbb{N} \Rightarrow \complement_{\mathbb{N}} B = A_{n+1} \in \tau \Rightarrow$ $B$ is closed. Hence, the closed sets in $(\mathbb{N}, \tau)$ are $\emptyset$, $\mathbb{N}$ and any set of the form $\{0, 1, ..., n\}$, where $n \in \mathbb{N}$.

(4) We have $V \in \mathcal{N}(2) \Leftrightarrow \exists n \in \mathbb{N}: \; 2 \in \underbrace{A_n}_{\in\tau} \subset V$. Hence, $\mathcal{N}(2) = \{A_0 = \mathbb{N}, A_1 = \{1, 2, ...\}, A_2 = \{2, 3, ...\}\}$. In the same way, $\mathcal{N}(3) = \{A_0 = \mathbb{N}, A_1 = \{1, 2, ...\}, A_2 = \{2, 3, ...\}, A_3 = \{3, 4, ...\}\}$. We see that $V \cap W \neq \emptyset$ for all $V \in \mathcal{N}(2)$ and $W \in \mathcal{N}(3)$, then $(\mathbb{N}, \tau)$ is not Hausdorff.

(5) Let $(\Omega_i)_{i\in I}$ be an open caver of $\mathbb{N}$, that is $\mathbb{N} \subset \bigcup\limits_{i\in I} \Omega_i$. Since $0 \in \mathbb{N}$ and $A_0 = \mathbb{N}$ is the unique open set of $\tau$ containing 0, there exists $i_0 \in I$ such that $\Omega_{i_0} = \mathbb{N}$. Thus, $(\Omega_i)_{i\in I}$ has a finite subcover, and $(\mathbb{N}, \tau)$ is compact.

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