2 Math
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Correction of Exam Algebra 3

Exo n°1 (6 pts)

1) (b) By definition, a diagonalizable matrix admit a basis of eigenvectors, hence it has n linearly independent eigenvectors.

2) (a) We have $f(v)=\lambda v$, and $f^2=I \Rightarrow f^2(v)=v \Rightarrow f(f(v))=1$
$\Rightarrow f(\lambda v)=v \Rightarrow \lambda f(v)=v \Rightarrow \lambda^2 v=v \Rightarrow (\lambda^2-1)v=0$,
Since $v\neq 0$, then $\lambda^2-1=0 \Rightarrow \lambda=-1,1$.

3) (a) We have $e^A = I + A + \frac{1}{2!}A^2 + \frac{1}{3!}A^3 + \cdots$
$A^2=-(2\pi)^2 I$, $A^3=-(2\pi)^2 A$, $A^4=(2\pi)^4 I$, ... Then
$$e^A = \left(1-\frac{1}{2!}(2\pi)^2+\frac{1}{4!}(2\pi)^4-\cdots\right)I + \left(1-\frac{1}{3!}(2\pi)^2-\cdots\right)A$$
$$=(\cos 2\pi)I + (\sin 2\pi)A = I.$$

Exo n°2 (9 pts)

1) $P_{A_\alpha}(\lambda)=(4-\lambda)(3-\lambda+\alpha)(3-\lambda-\alpha)$.

2) If $\alpha=0$, $\lambda_1=4\ (k_1=1)$, $\lambda_2=3\ (k_2=2)$. If $\alpha=\pm1$, $\lambda_1=4$ $(k_1=2)$, $\lambda_2=2\ (k_2=1)$. If $\alpha\in\mathbb{R}-\{0,1,-1\}$, $\lambda_1=4\ (k_1=1)$, $\lambda_2=3+\alpha\ (k_2=1)$, $\lambda_3=3-\alpha\ (k_3=1)$.

3) $\alpha=0$: $P_{A_0}(\lambda)=(4-\lambda)(3-\lambda)^2$, $E_4=\left\langle\begin{pmatrix}0\\1\\0\end{pmatrix}\right\rangle$, $E_3=\left\langle\begin{pmatrix}0\\-2\\1\end{pmatrix}\right\rangle$

4) $\dim E_3\neq 2$, then A_0 is not diagonalizable.

5) $\tilde{v}_1=v_1=\begin{pmatrix}0\\1\\0\end{pmatrix}$, $\tilde{v}_2=v_2=\begin{pmatrix}0\\-2\\1\end{pmatrix}$, $(A-3I)\tilde{v}_3=\tilde{v}_2$
$\Rightarrow \tilde{v}_3=\begin{pmatrix}-1\\0\\0\end{pmatrix}$, $\tilde{P}=\begin{pmatrix}0&0&-1\\1&-2&0\\0&1&0\end{pmatrix}$, $\tilde{P}^{-1}=\begin{pmatrix}0&1&2\\0&0&1\\-1&0&0\end{pmatrix}$.

6) $J=\begin{pmatrix}4&0&0\\0&3&1\\0&0&3\end{pmatrix}$, $e^{tJ}=\begin{pmatrix}e^{4t}&0&0\\0&e^{3t}&te^{3t}\\0&0&e^{3t}\end{pmatrix}$

7) $e^{tA} = \tilde{P} e^{tJ} \tilde{P}^{-1} = \begin{pmatrix} e^{3t} & 0 & 0 \\ 2(e^{4t}-e^{3t}) & e^{4t} & 2(e^{4t}-e^{3t}) \\ e^{3t}-e^{4t} & 0 & e^{3t} \end{pmatrix}$

Ex n° 3 : (5pts)

1) We have $P_A(\lambda) = (-2-\lambda)^3$. By Cayley-Hamilton theorem, we get $(-2I-A)^3 = 0$. Then $(A+2I)^3 = 0$.

We have $e^{tA} = e^{t(A+2I-2I)} = e^{t(A+2I)} e^{-2tI}$.

$e^{t(A+2I)} = I + t(A+2I) + \frac{t^2}{2!}(A+2I)^2 = \begin{pmatrix} 1 & 3t+3t^2 & 4t+9t^2 \\ 0 & 1+2t+2t^2 & 6t+6t^2 \\ 0 & 0 & 1 \end{pmatrix}$,

and $e^{-2tI} = \begin{pmatrix} e^{-2t} & 0 & 0 \\ 0 & e^{-2t} & 0 \\ 0 & 0 & e^{-2t} \end{pmatrix} = e^{-2t} I$. Then,

$$\boxed{e^{tA} = e^{-2t} \begin{pmatrix} 1 & 3t+3t^2 & 4t+9t^2 \\ 0 & 1+2t+2t^2 & 6t+6t^2 \\ 0 & 0 & 1 \end{pmatrix}},$$

2) We know that $Y(t) = e^{(t-t_0)A} Y(t_0)$. Then

$Y(t) = e^{(t+1)A} Y(-1)$

$= e^{-2(t+1)} \begin{pmatrix} 1 & 3t^2+9t+6 & 9t^2+22t+13 \\ 0 & 2t^2+6t+7 & 6t^2+18t+12 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \vdots \\ 2 \\ 2 \end{pmatrix}$

$$\boxed{Y(t) = \begin{pmatrix} 24231t^2 + 62753t + 40488 \\ 16194t^2 + 48582t + 38463 \\ 2024 \end{pmatrix}}$$