University of El Oued | Academic year 2025/2026
Faculty of Exact Sciences | Level : L2
Department of Mathematics | Module : Introduction to topology

Exam

Sunday, 25 January 2026

Exercise 1. [7 points]

Consider $\mathbb{R}$ equipped with the distance

$$\forall x, y \in \mathbb{R}: \quad d(x,y) = |e^x - e^y|.$$

(1) Find $B(0,1)$ the open ball of center 0 and radius 1.

(2) Let $A =]-\infty, 0]$. Prove that A is bounded and closed in $(\mathbb{R}, d)$.

(3) Prove that $(\mathbb{R}, d)$ is not complete (indication : use the sequence $x_n = -n$, $n \in \mathbb{N}$).

(4) Prove that A is not compact.

Exercise 2. [5 points]

(1) Consider

$$\forall x, y \in \mathbb{R}: \quad \delta(x,y) = \ln(1 + |x-y|).$$

Prove that δ is a distance on $\mathbb{R}$.

(2) Let $E = C([a,b], \mathbb{R})$ be the vector space of continuous functions on the closed and bounded interval $[a,b]$ equipped with the uniform norm $\|f\|_\infty = \sup\limits_{x \in [a,b]} |f(x)|$, $f \in E$. Consider $\varphi : (E, \|.\|_\infty) \to (\mathbb{R}, |.|)$, $f \mapsto \varphi(f) = \displaystyle\int_a^b f(x)\, dx - f\Big(\frac{a+b}{2}\Big)$.

Prove that φ is linear and continuous.

Exercise 3. [8 points]

Let $n \in \mathbb{N}$, $A_n = \{n, n+1, n+2, ...\}$ and

$$\begin{aligned} \tau &= \{\emptyset\} \cup \{A_n \,|\, n \in \mathbb{N}\} \\ &= \{\emptyset\} \cup \{A_0, A_1, A_2, ...\}. \end{aligned}$$

(1) Show that τ is a topology on $\mathbb{N}$.

(2) Let $B = \{1, 3, 5, 7, ...\}$. Find $\mathring{B}$ in $(\mathbb{N}, \tau)$.

(3) Find the closed sets in $(\mathbb{N}, \tau)$.

(4) Determine $\mathcal{N}(2)$ and $\mathcal{N}(3)$. Deduce that $(\mathbb{N}, \tau)$ is not Hausdorff.

(5) Prove that $(\mathbb{N}, \tau)$ is compact.

Exercise 1. (1) We have

$$B(0,1)=\{x\in\mathbb{R}|\,d(x,0)<1\}=\{x\in\mathbb{R}|\,|e^x-\underbrace{e^0}_{=1}|<1\}$$
$$=\{x\in]-\infty,0[|\,1-e^x<1\}\cup\{x\in[0,\infty[|\,e^x-1<1\}$$
$$=\{x\in]-\infty,0[|\,0<e^x\}\cup\{x\in[0,\infty[|\,x<\ln 2\}=]-\infty,0[\cup[0,\ln 2[=]-\infty,\ln 2[.$$

(2) A is bounded. We see that $A=]-\infty,0]\subset]-\infty,\ln 2[=B(0,1)$. Hence, A is bounded in $(\mathbb{R},d)$. A is closed. Let $(x_n)_{n\in\mathbb{N}}\subset A$ be a sequence that converges in $(\mathbb{R},d)$ to some limit $l\in\mathbb{R}$, that is $\lim\limits_{n\to\infty} d(x_n,l)=\lim\limits_{n\to\infty}|e^{x_n}-e^l|=0$. This gives $\lim\limits_{n\to\infty} e^{x_n}=e^l$ or $\lim\limits_{n\to\infty} x_n=l$. Since $x_n\leq 0$ for all $n\in\mathbb{N}$, it follows that $l=\lim\limits_{n\to\infty} x_n\leq 0$. Hence $l\in A$, and so A is closed in $(\mathbb{R},d)$.
(3) The sequence $x_n=-n$ is Cauchy in $(\mathbb{R},d)$ because $\lim\limits_{p,q\to\infty} d(x_p,x_q)=\lim\limits_{p,q\to\infty}|e^{-p}-e^{-q}|=0$. By absurd, suppose that $(-n)_{n\in\mathbb{N}}$ converges in $(\mathbb{R},d)$ to some limit l. Namely, $\lim\limits_{n\to\infty} d(-n,l)=\lim\limits_{n\to\infty}|e^{-n}-e^l|=0$. This gives $\underbrace{\lim\limits_{n\to\infty} e^{-n}}_{=0}=e^l>0$, which is absurd. Hence, the sequence $(-n)_{n\in\mathbb{N}}$ is divergent in $(\mathbb{R},d)$, and so $(\mathbb{R},d)$ is not complete.
(4) We see that $x_n=-n\in A$ for all $n\in\mathbb{N}$. By absurd, suppose that $(-n)_{n\in\mathbb{N}}$ has a subsequence $(-\varphi(n))_{n\in\mathbb{N}}$, where $\varphi:\mathbb{N}\to\mathbb{N}$ is a strictly increasing map, converging to some limit l. This leads to $\underbrace{\lim\limits_{n\to\infty} e^{-\varphi(n)}}_{=0}=e^l>0$, which is impossible. Hence, any subsequence of $(-n)_{n\in\mathbb{N}}$ is divergent, and so A is not compact in $(\mathbb{R},d)$.

Exercise 2.
(1) (i) $\delta(x,y)=0\Leftrightarrow\ln(1+|x-y|)=0\Leftrightarrow 1+|x-y|=1\Leftrightarrow|x-y|=0\Leftrightarrow x=y$.
(ii) $\forall x,y\in\mathbb{R}:\quad\delta(x,y)=\ln(1+|x-y|)=\ln(1+|y-x|)=\delta(y,x)$, then δ is symmetry.
(iii) Triangle inequality. Let $x,y,z\in\mathbb{R}$. We have

$$\begin{aligned}\delta(x,y)\leq\delta(x,z)+\delta(z,y)&\Leftrightarrow\ln(1+|x-y|)\leq\ln(1+|x-z|)+\ln(1+|z-y|)\\&\Leftrightarrow e^{\ln(1+|x-y|)}\leq e^{\ln(1+|x-z|)+\ln(1+|z-y|)}\\&\Leftrightarrow e^{\ln(1+|x-y|)}\leq e^{\ln(1+|x-z|)}e^{\ln(1+|z-y|)}\\&\Leftrightarrow 1+|x-y|\leq(1+|x-z|)(1+|z-y|)\\&\Leftrightarrow\underbrace{|x-y|\leq|x-z|+|z-y|+|x-z||z-y|}_{\text{it is true}}.\end{aligned}$$

Hence, the triangle inequality holds. From (i), (ii) and (iii), δ is a distance on $\mathbb{R}$.
(2) φ is linear. For all $f,g\in E$ and $\lambda\in\mathbb{K}$, we have

$$\begin{aligned}\varphi(\lambda f+g)&=\int_a^b(\lambda f+g)(x)\,dx-(\lambda f+g)\Big(\frac{a+b}{2}\Big)\\&=\int_a^b(\lambda f)(x)+g(x)\,dx-(\lambda f)\Big(\frac{a+b}{2}\Big)-g\Big(\frac{a+b}{2}\Big)\\&=\int_a^b\lambda f(x)\,dx+\int_a^b g(x)\,dx-\lambda f\Big(\frac{a+b}{2}\Big)-g\Big(\frac{a+b}{2}\Big)\\&=\lambda\int_a^b f(x)\,dx+\int_a^b g(x)\,dx-\lambda f\Big(\frac{a+b}{2}\Big)-g\Big(\frac{a+b}{2}\Big)\\&=\lambda\Big[\int_a^b f(x)\,dx-f\Big(\frac{a+b}{2}\Big)\Big]+\int_a^b g(x)\,dx-g\Big(\frac{a+b}{2}\Big)\\&=\lambda\varphi(f)+\varphi(g).\end{aligned}$$

So, φ is linear.

<u>Continuity of φ</u>. One can write

$$\begin{aligned}\forall f\in E:\quad |\varphi(f)| &= \Big|\int_a^b f(x)\,dx - f\Big(\frac{a+b}{2}\Big)\Big| \\ &\le \Big|\int_a^b f(x)\,dx\Big| + \Big|f\Big(\frac{a+b}{2}\Big)\Big| \\ &\le \int_a^b |f(x)|\,dx + \Big|f\Big(\frac{a+b}{2}\Big)\Big| \\ &\le \sup_{x\in[a,b]}|f(x)|\int_a^b dx + \sup_{x\in[a,b]}|f(x)| \\ &= (b-a)\|f\|_\infty + \|f\|_\infty \\ &\le \underbrace{\max(b-a,1)}_{C}\,\|f\|_\infty \text{..............................}(*)\end{aligned}$$

Since φ is linear, it follows from $(*)$ that φ is continuous on E.

<u>Exercice 3.</u>

(1) (i) $\emptyset\in\tau$ and $\mathbb{N}=A_0\in\tau$.

(ii) Let $B_1, B_2\in\tau$. We distinguish two cases :

• $B_1=\emptyset$ or $B_2=\emptyset$, then $B_1\cap B_2=\emptyset\in\tau$.

• $B_1\neq\emptyset$ and $B_2\neq\emptyset$, then $B_1=A_{n_1}$ and $B_2=A_{n_2}$ with $n_1,n_2\in\mathbb{N}$. Setting $n_3=\max(n_1,n_2)$, we get $B_1\cap B_2=A_{n_3}\in\tau$.

(iii) Let $(B_i)_{i\in I}\subset\tau$. We distinguish two cases :

$*$ $\forall i\in I:\ B_i=\emptyset$, then $\bigcup\limits_{i\in I}B_i=\emptyset\in\tau$.

$*$ There exists $i\in I$ such that $B_i\neq\emptyset$. Therefore, $B_i=A_{n_i}$ for some $n_i\in\mathbb{N}$. Setting $m=\min\{n_i|\,B_i=A_{n_i}, i\in I, n_i\in\mathbb{N}\}$, we find $\bigcup\limits_{i\in I}B_i=A_m\in\tau$.

From (i), (ii) and (iii), τ is a topology on $\mathbb{N}$.

(2) If $m\in B=\{1,2,3,7,...\}$, then $m\notin\mathring{B}$ because there do not exist $n\in\mathbb{N}$ such that $m\in\underbrace{A_n}_{\in\tau}\subset B$. Hence, $\mathring{B}=\emptyset$.

(3) B is closed $\Rightarrow \complement_{\mathbb{N}}B\in\tau \Rightarrow \complement_{\mathbb{N}}B=\emptyset$ or $\complement_{\mathbb{N}}B=A_n$, $n\in\mathbb{N}\Rightarrow B=\mathbb{N}$ or $B=\emptyset$ or $B=\{0,1,...,n\}$, $n\in\mathbb{N}$. If $B=\{0,1,...,n\}$ for some $n\in\mathbb{N}\Rightarrow \complement_{\mathbb{N}}B=A_{n+1}\in\tau\Rightarrow$ B is closed. Hence, the closed sets in $(\mathbb{N},\tau)$ are $\emptyset$, $\mathbb{N}$ and any set of the form $\{0,1,...,n\}$, where $n\in\mathbb{N}$.

(4) We have $V\in\mathcal{N}(2)\Leftrightarrow\exists n\in\mathbb{N}:\ 2\in\underbrace{A_n}_{\in\tau}\subset V$. Hence, $\mathcal{N}(2)=\{A_0=\mathbb{N}, A_1=\{1,2,...\}, A_2=\{2,3,...\}\}$. In the same way, $\mathcal{N}(3)=\{A_0=\mathbb{N}, A_1=\{1,2,...\}, A_2=\{2,3,...\}, A_3=\{3,4,...\}\}$. We see that $V\cap W\neq\emptyset$ for all $V\in\mathcal{N}(2)$ and $W\in\mathcal{N}(3)$, then $(\mathbb{N},\tau)$ is not Hausdorff.

(5) Let $(\Omega_i)_{i\in I}$ be an open caver of $\mathbb{N}$, that is $\mathbb{N}\subset\bigcup\limits_{i\in I}\Omega_i$. Since $0\in\mathbb{N}$ and $A_0=\mathbb{N}$ is the unique open set of τ containing 0, there exists $i_0\in I$ such that $\Omega_{i_0}=\mathbb{N}$. Thus, $(\Omega_i)_{i\in I}$ has a finite subcover, and $(\mathbb{N},\tau)$ is compact.

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