University of El-Oued

Exact Sciences Faculty **Academic Year: 2025 / 2026**

Computer Science Department **Duration: 1h30**

Solution of Exam: Analysis 01

Full name: ./ **Group**: /

Exercise 01: Let $z = cos\left(\frac{2\pi}{7}\right) + isin\left(\frac{2\pi}{7}\right)$. Calculate z^7 and deduce the value of $S = \sum_{k=0}^{k=6} z^k$.

We have

$$z^7 = cos\left(\frac{2\pi}{7} \times 7\right) + isin\left(\frac{2\pi}{7} \times 7\right) = cos(2\pi) + isin(2\pi) = 1 \ldots\ldots\ldots (01.5p)$$

And

$$S = \sum_{k=0}^{k=6} z^k = \frac{1-z^7}{1-z} = \frac{1-1}{1-z} = \frac{0}{1-z} = 0 \ldots\ldots\ldots (01.5p)$$

Exercise 02: Let (u_n) be a real sequence defined by

$$\begin{cases} u_0 = 0 \\ u_{n+1} = \frac{u_n{}^2 + u_n + 1}{u_n + 2}, \; n \in \mathbb{N} \end{cases}$$

1. Prove that $\forall n \in \mathbb{N}: 0 \leq u_n < 1$.

We prove by induction. It's clear that $0 \leq u_0 = 0 < 1 \ldots\ldots$(01p)

Assume that $0 \leq u_n < 1$.

Let $f(x) = \frac{x^2+x+1}{x+2}$. We have f is strictly increasing on $[0,1[$ …..(01p)

then

$$f(0) \leq f(u_n) < f(1)$$

So

$$0 \leq \frac{1}{2} \leq \frac{u_n{}^2 + u_n + 1}{u_n + 2} = u_{n+1} < \frac{3}{3} = 1 \ldots\ldots\ldots (01.5p)$$

therefore

$$0 \leq u_{n+1} < 1$$

Thus $\forall n \in \mathbb{N}: 0 \leq u_n < 1$.

2. Show that (u_n) is strictly increasing, what do you conclude?

We have

$$\forall n \in \mathbb{N}: u_{n+1} - u_n = \frac{u_n{}^2 + u_n + 1}{u_n + 2} - u_n = \frac{1-u_n}{u_n+2} > 0 \ldots\ldots\ldots (01.5p)$$

Then (u_n) is strictly increasing.

Since (u_n) is strictly increasing and bounded above by 1, therefore, it is convergent and its limit is l such that

$$\frac{l^2+l+1}{l+2} = l \Rightarrow l = 1 \ldots\ldots\ldots (01p)$$

Exercise 03: 1. Let $f(x) = \begin{cases} 1 - x^2 ln|x|, & x \neq 0 \\ 1, & x = 0 \end{cases}$. Prove that $f \in C^1(\mathbb{R})$.

It's clear that f is continuous on $\mathbb{R}^*$, we have

$$\lim_{x \to 0} f(x) = \lim_{x \to 0} (1 - x^2 ln|x|) = 1 = f(0)$$

so f is continuous at 0 then f is continuous on $\mathbb{R}$. …….(01p)

We have also f is differentiable on $\mathbb{R}$ because

$$\forall x \in \mathbb{R}^*: \; f^{/}(x) = -2xln|x| - x \ldots\ldots (0.5p)$$

and

$$\lim_{x \to 0} \frac{f(x) - f(0)}{x - 0} = \lim_{x \to 0} \frac{-x^2 ln|x|}{x} = \lim_{x \to 0} -xln|x| = 0 = f^{/}(0) \ldots\ldots (0.5p)$$

We have $f^{/}$ is continuous on $\mathbb{R}^*$ and

$$\lim_{x \to 0} f^{/}(x) = \lim_{x \to 0} (-2xln|x| - x) = 0 = f^{/}(0) \ldots\ldots (0.5p)$$

Then $f^{/}$ is continuous at 0.

Therefore $f^{/}$ is continuous on $\mathbb{R}$.......(0.5p)

Finally, we conclude that $f \in C^1(\mathbb{R})$.

2. Let $f(x) = x^3 + 3x + 1$, calculate $(f^{-1})^{/}(1)$.

We know that

$$(f^{-1})^{/}(1) = \frac{1}{f^{/}(f^{-1}(1))}.......(0.5p)$$

We have

$$f^{-1}(1) = \alpha \Leftrightarrow f(\alpha) = 1 \Leftrightarrow x^3 + 3x + 1 = 1 \Leftrightarrow x^3 + 3x = 0 \Leftrightarrow \alpha(\alpha^2 + 3) = 0 \Leftrightarrow \alpha = 0....(0.5p)$$

So

$$f^{-1}(1) = 0 \;\; and \;\; f^{/}\big(f^{-1}(1)\big) = f^{/}(0) = 3 \times 0^2 + 3 = 3....(0.5p)$$

thus

$$(f^{-1})^{/}(1) = \frac{1}{3}.......(0.5p)$$

Exercise 04: Let $f:]1, +\infty[\to \mathbb{R},\; f(x) = arcsin(1/x)$.
1. Determine the limits of f and study the direction of change of f.
We have

$$\lim_{x\to 1} f(x) = \lim_{x\to 1} arcsin(1/x) = arcsin(1) = \frac{\pi}{2}......(01p)$$

$$\lim_{x\to +\infty} f(x) = \lim_{x\to +\infty} arcsin(1/x) = arcsin(0) = 0......(01p)$$

We have f is differentiable on $]1, +\infty[$ and

$$\forall x \in]1, +\infty[:\; f^{/}(x) = -\frac{1}{x^2}.\frac{1}{\sqrt{1-x^2}} < 0......(0.5p)$$

Therefore f is strictly decreasing......(0.5p)

2. Prove that $f(x) = \frac{\pi}{4}$ has a unique solution on $]1, +\infty[$ and trace the graph of f.

We have

$$f(x) = \frac{\pi}{4} \Leftrightarrow \underbrace{f(x) - \frac{\pi}{4}}_{g(x)} = 0$$

The function g is continuous and strictly decreasing on $]1, +\infty[$. So, by using the intermediate value theorem, we find that the equation $g(x) = 0 \Leftrightarrow f(x) = \frac{\pi}{4}$ has a unique solution on $]1, +\infty[$......(01.5p)

The graph of f.......(01.5p)